McMule for MUonE

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fixed-order NNLO QED framework

- provided: matrix elements by us or others
- output: physical cross section for any physical observable
- McMule: phase space generation, subtraction, stabilisation, integration, etc.

- all leptonic $2 \to 2$ processes in QED at NNLO (+ a few others)
- integrator & generator
- user defines cuts through arbitrary function that is loaded at run time

Get the code here: `https://mule-tools.gitlab.io`
Read the docs here: `https://mcmule.readthedocs.io`

McMule

process	experiment	physics motivation	order
$e\mu \to e\mu$	MUonE	HVP to $(g-2)_\mu$	NNLO+
$\ell p \to \ell p$	AMBER, PRad, ULQ2, ...	proton radius and weak charge	NNLO
$eN \to eN$	PRad, ULQ2	background	NNLO−
$e^-e^- \to e^-e^-$	Prad 2	normalisation	NNLO
	MOLLER, ...	$\sin^2\theta_W$ at low Q^2	
$e^+e^- \to e^+e^-$	any e^+e^- collider	luminosity measurement	NNLO
$ee \to \ell\ell$	VEPP, BES, Daphne, ...	R-ratio	NNLO+
	Belle	τ properties	
$ee \to \gamma\gamma$	Daphne	dark searches	NNLO−
	any e^+e^- collider	luminosity measurement	
$e\nu \to e\nu$	DUNE	flux & $\sin^2\theta_W$	NNLO−
$\mu \to \nu\bar{\nu}e$	MEG	ALP searches	NNLO+
	DUNE	beam-line profiling	
$\mu \to \nu\bar{\nu}e\gamma$	MEG, Mu3e, Pioneer	background	NLO
$\mu \to \nu\bar{\nu}eee$	MEG, Mu3e	background	NLO
$ee \to \pi\pi$	VEPP, BES, Daphne, ...	R-ratio	NLO
$ee \to \ell\ell\gamma$	VEPP, BES, Daphne, ...	R-ratio	NNLO−

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$e^+e^- \to e^+e^-$	any e^+e^- cc	neasurement	NNLO
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$e\nu \to e\nu$	DUNE	$_W$	NNLO−
$\mu \to \nu\bar\nu e$	MEG		NNLO+
	DUNE	ling	
$\mu \to \nu\bar\nu e\gamma$	MEG, Mu3e, Pioneer	background	NLO
$\mu \to \nu\bar\nu eee$	MEG, Mu3e	background	NLO
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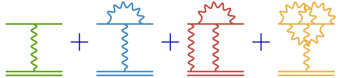
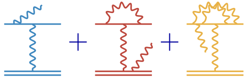
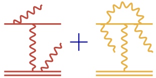
$ee \to \ell\ell\gamma$	VEPP, BES, Daphne, ...	R-ratio	NNLO−

goal: world domination

anatomy of an NNNLO calculation

$\sigma = \int \mathrm{d}\Phi_2 \Bigg|$  $+ \dots \Bigg|^2$

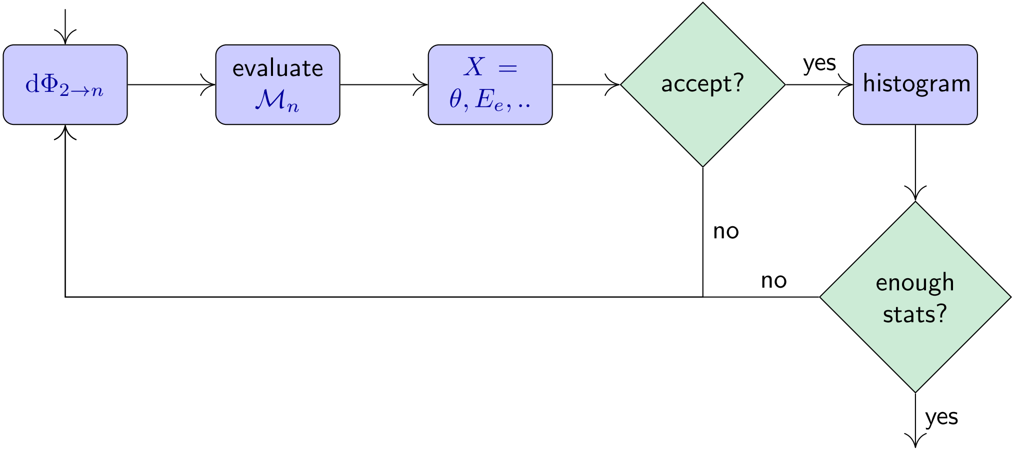
$+ \int \mathrm{d}\Phi_3 \Bigg|$  $+ \dots \Bigg|^2$

$+ \int \mathrm{d}\Phi_4 \Bigg|$  $+ \dots \Bigg|^2$

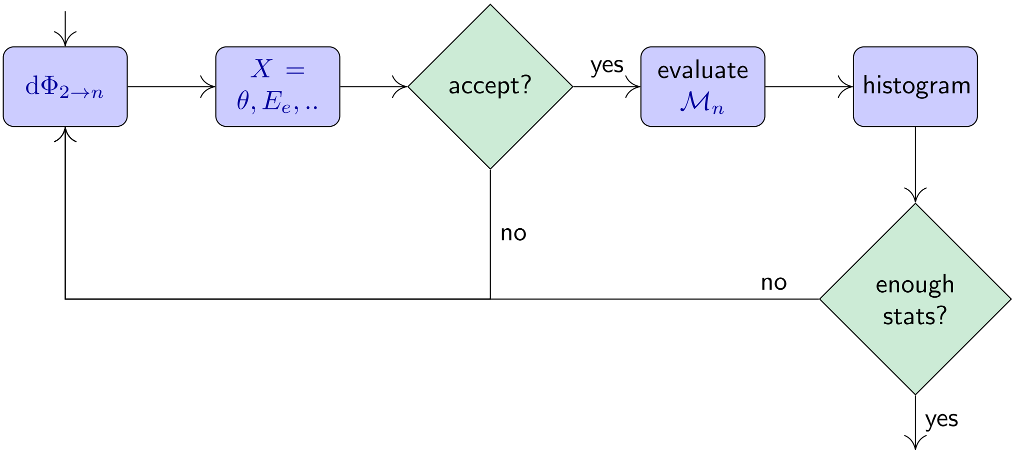
$+ \int \mathrm{d}\Phi_5 \Bigg|$  $+ \dots \Bigg|^2$

$+ \dots$

a simple analysis assuming X is difficult: a generator



a simple analysis assuming $\mathcal{M}_n$ is difficult: an integrator



why is the theory pipeline different?

- assumption: for a pure theory calculation, X is trival to calcualte

```
q3lab = boost_rf(q1,q3)
q4lab = boost_rf(q1,q4)
theta_e = acos(cos_th(ez,q3lab))
theta_m = acos(cos_th(ez,q4lab))

gmu = (q2lab(4)*me+mm**2)/(q2lab(4)*me+me**2)
gam = (q2lab(4)+me)/sqrt(scms)
theta_m_el = atan(2*tan(theta_e)/((1+gam**2*tan(theta_e)**2)*(1+gmu)-2))
bdev = theta_m/theta_m_el

if(Ee<1000.) pass_cut = .false.
if(theta_m<0.3E-3) pass_cut = .false.
!band cut
if(bdev<0.9) pass_cut = .false.
if(bdev>1.1) pass_cut = .false.
```

- but $\mathcal{M}_n$ is really slow

why is the theory pipeline different?

- assumption: for a pure theory calculation, X is trival to calcualte
- but some parts of $\mathcal{M}_n$ are very slow
- $\displaystyle\int \mathrm{d}\Phi_{n+1}$ is divergent!

what does this mean?

- decide early whether $\mathcal{M}_n$ needs to be calculated
- $\mathcal{M}_n = \underbrace{\mathcal{M}_n^{(0)}}_{\text{fast and large}} + \underbrace{\mathcal{M}_n^{(1)}}_{\text{okay-ish and large-ish}} + \underbrace{\mathcal{M}_n^{(2)}}_{\text{unbelievably slow but small}} + \cdots$
- this allowed us to do the first full* NNLO calculation of $e\mu \to e\mu$ [Broggio, Engel, Ferroglia, Mandal, Mastrolia, Rocco, Ronca, Signer, Torres Bobadilla, Zoller, YU 22]
- require a subtraction

$\Rightarrow$ weights have no probabilisic intepretation

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garden-hose approach

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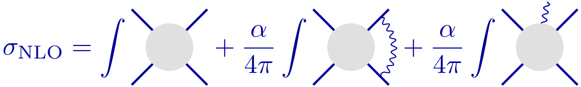
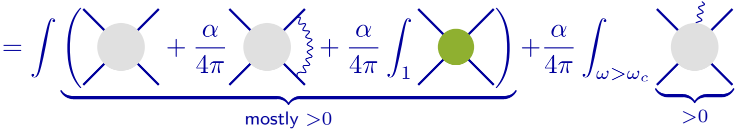
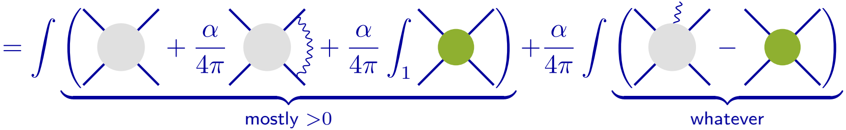
... at NLO for simplicity



- slicing: fairly few negative weights but numerically construct $\log \omega_c$



- subtraction: easier integration but lots and lots of negative weights ($\mathcal{O}(5\%)$ at NLO, more at NNLO)



my proposal: build two streams

best theory

- early access to improvements
- currently: full NNLO
- future: N^3LO, NLP soft resummation
- better handling of hadronic effects
- better numerical stability & efficiency

good experiment

- less complicated detector simulation $\mathcal{O}(\text{ms/ev})$
- potentially involving interpolations and other sources of uncertainties
- can be done with GEANT4

best experiment

- full-blown detector simulation
- full event generation from McMule (WIP)
- no modelling uncertainties

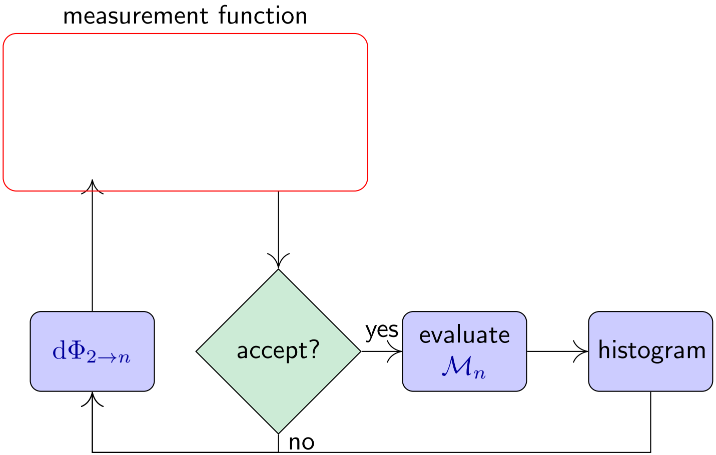
good theory

- "only" NNLO
- no early access to features
- significantly increased runtime & storage requirement

two examples in $ep \to ep$

ULQ2 (Tohoku University), PRad II (JLab)

- PRad II is adopting McMule into their analysis flow



gitlab.comyannickulrich/simple_geant4_package

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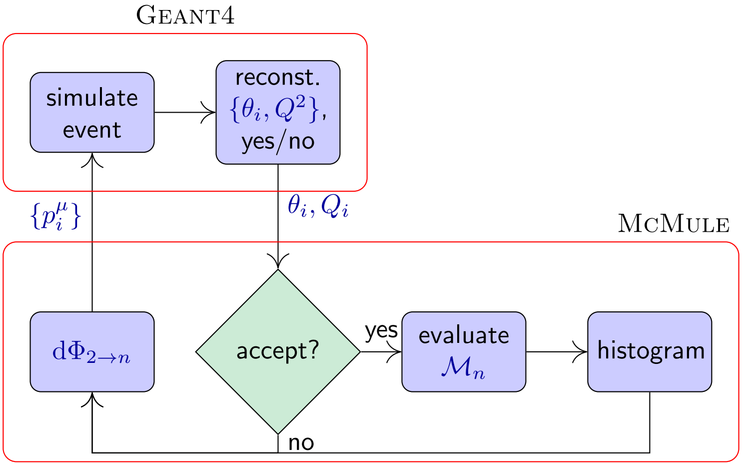

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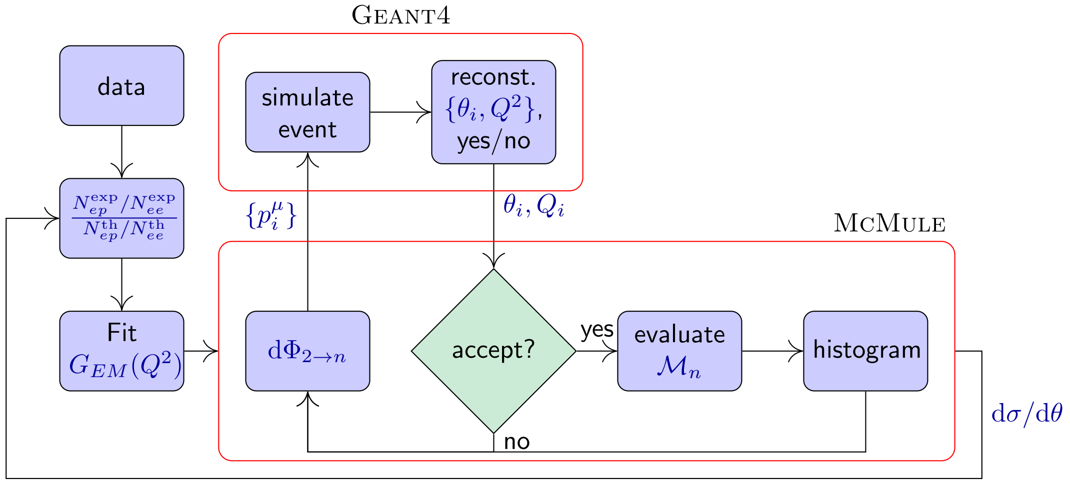

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- buidling & validating generator
- generator efficiency $\sim (w_{\rm max} - w_{\rm min})/\langle w\rangle$

Order	ξ_c	N	r	$[w_{\rm min}, w_{\rm max}]/\langle w\rangle$	feature
LO	n/a	99.7M	0.0%	$[0.03, 7.7]$	
NLO	0.1	220M	0.5%	$[-3.6, 3.6]\times 10^7$	
	1.0	216M	3.5%	$[-2.1, 2.1]\times 10^6$	
NNLO	0.1	21.1G	0.7%	$[-8.6, 8.6]\times 10^7$	
	1.0	19.8G	9.2%	$[-2.3, 2.3]\times 10^8$	

- buidling & validating ger
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Order	ξ_c	N	$]/\langle w\rangle$	**feature**
LO	n/a	9		
NLO	0.1	2	10^7	
	1.0	2	10^6	
NNLO	0.1	2	10^7	
	1.0	1	10^8	

can we get both?

two observations

1. cross section $\sigma = \int_{\mathcal{C}} \mathrm{d}\sigma > 0$, irregardless of the size of integration region $\mathcal{C}$
2. experiments have a finite resolution

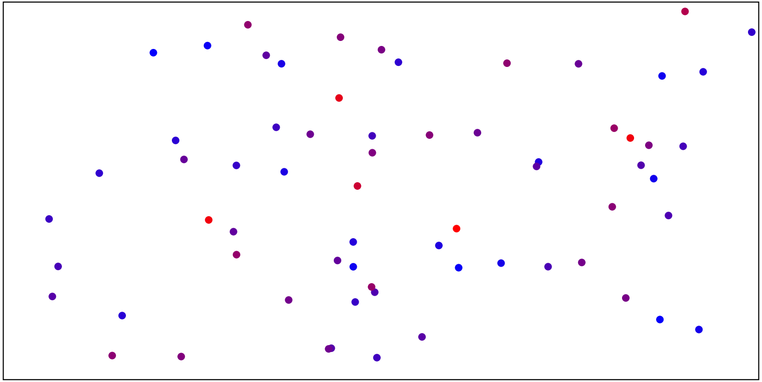
   (we already knew that because we can't see soft photons)

algorithm to remove negative weights [Andersen, Maier 21]

- pick an event with $w_i < 0$
- find nearby events until $\sum_{i\in\mathcal{C}} w_i > 0$
- if $\mathcal{C}$ gets too big (events become resolvable), abort (or add more events)
- else $w_i \to \frac{\sum_{j\in\mathcal{C}} w_j}{\sum_{j\in\mathcal{C}} |w_j|} w_i$

we can remove negative weights without biasing physical observables!

example



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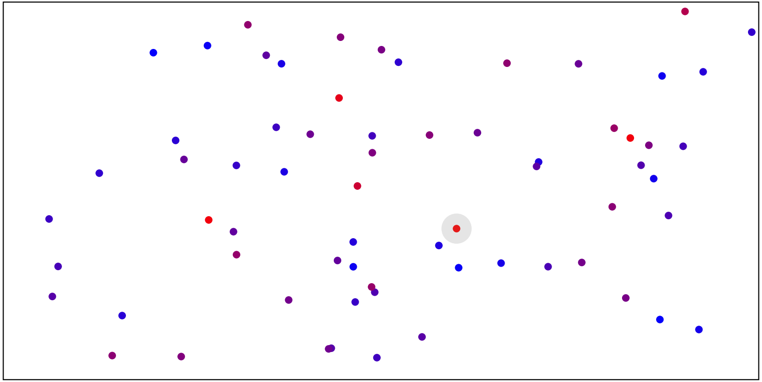

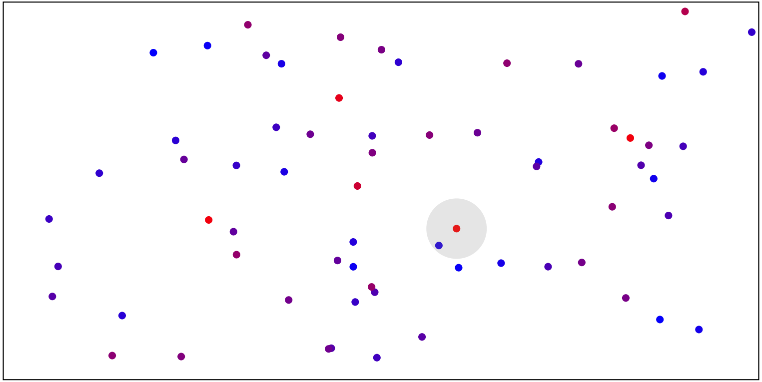

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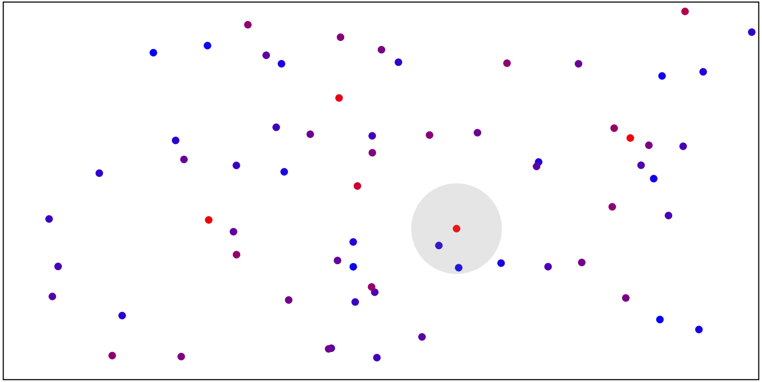

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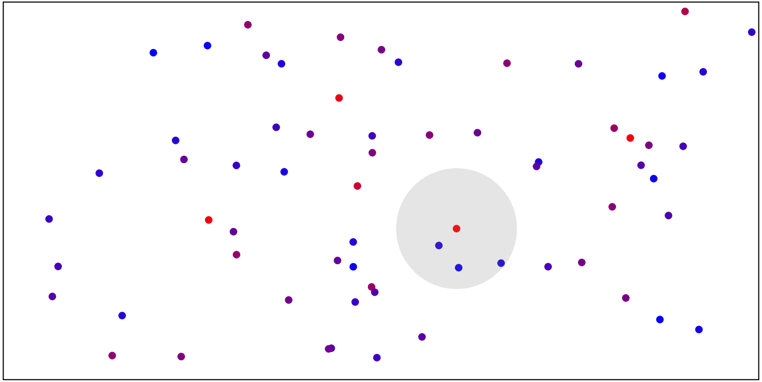

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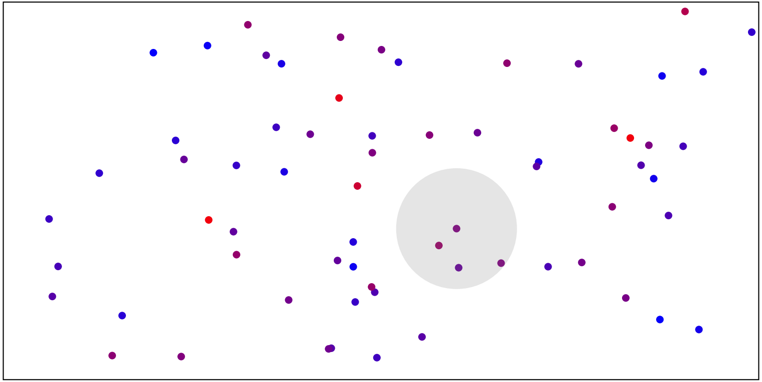

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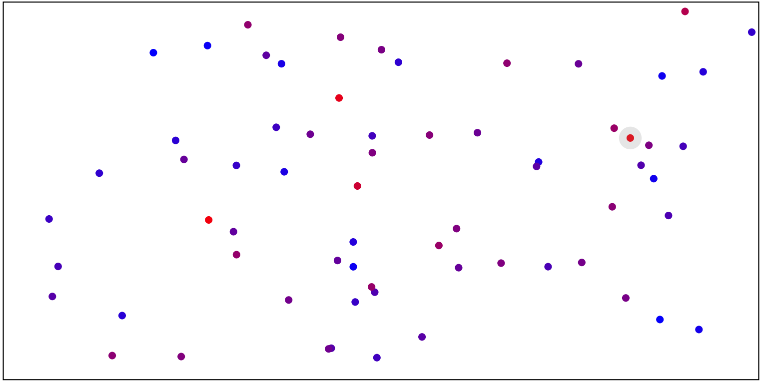

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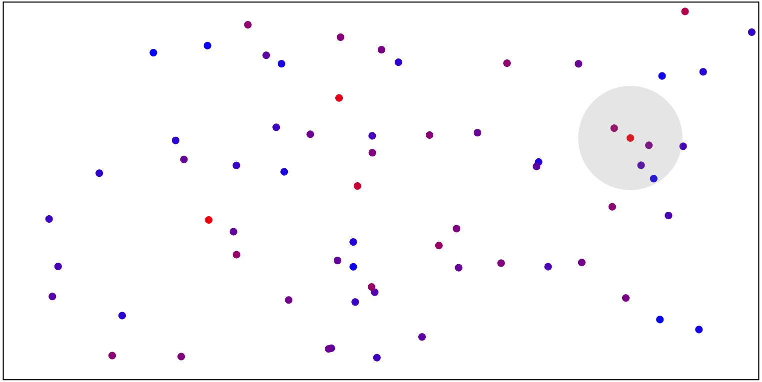

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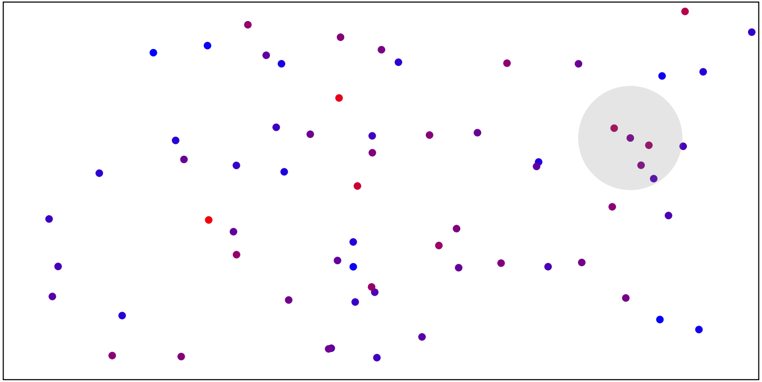

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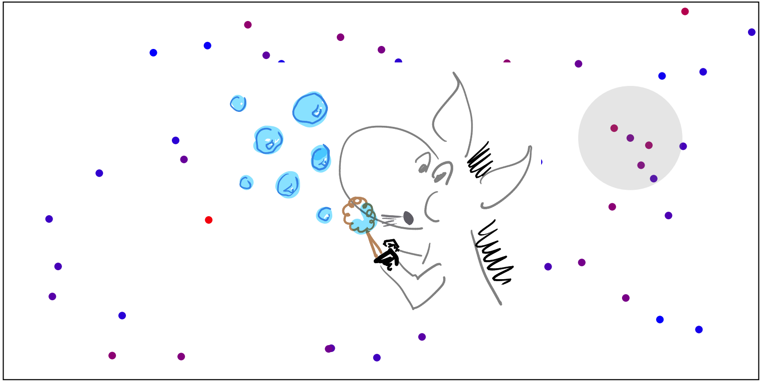
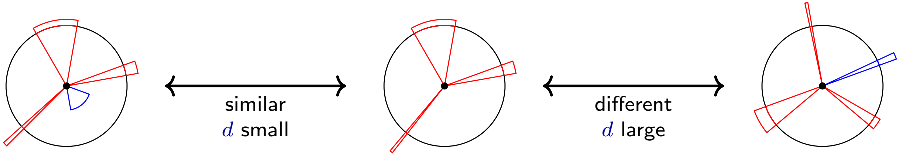


what does nearby mean?

we need to define a metric in event space $d(e_1, e_2) \geq 0$

- doesn't really matter how we do this as long as IR safe
  (events with soft photons are near each other)
- ideally: events that look similar are closer to each other than those that don't



- e.g. $d(\mathcal{E}^{(1)}, \mathcal{E}^{(2)}) = \sqrt{\left|\frac{\theta_e^{(1)} - \theta_e^{(2)}}{10\,\mathrm{mrad}}\right|^2 + \left|\frac{\theta_\mu^{(1)} - \theta_\mu^{(2)}}{1\,\mathrm{mrad}}\right|^2}$

- buidling & validating generator
- generator efficiency $\sim (w_{\text{max}} - w_{\text{min}})/\langle w\rangle$

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	1.0	216M	3.5%	$[-2.1, 2.1]\times 10^6$	
			0.3%	$[-7.0, 26]\times 10^3$	cres
NNLO	0.1	21.1G	0.7%	$[-8.6, 8.6]\times 10^7$	
		3.6G	0.1%	$[-6.9, 6.9]\times 10^5$	cres
	1.0	19.8G	9.2%	$[-2.3, 2.3]\times 10^8$	
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		51M	0	$[+0.0, 1.6] \times 10^4$	subs
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- generator efficiency $\sim (w \quad - w \quad)/\langle w\rangle$

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NLO	0.1	2			10^7	
					0^3	cres
		5			10^4	subs
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NNLO	0.1	2			10^7	
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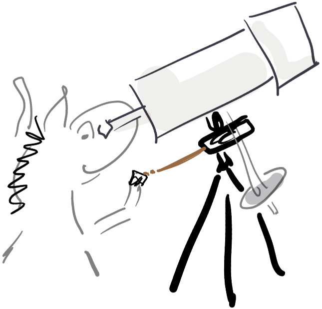
summary

- McMule is a (mostly $2 \to 2$) NNLO code for QED
- event generation at $2 \to 2$ NNLO works but leads to many negative weights
- cell resampling & subsampling might be able to (all but) elimate this problem
- but event generator will always lag behind integrator

$\Rightarrow$ experiments should develop fast simulations



outlook

- finish & test generator
- go beyond $2 \to 2$ NNLO: $2 \to 3$ NNLO, $2 \to 2$ N^{3}LO, resummation

f.l.t.r.: S.Kollatzsch (Zurich & PSI), A.Signer (Zurich & PSI), V.Sharkovska (Zurich & PSI), S.Gündogdu (Zurich & PSI), D. Moreno (PSI), A.Coutinho (IFIC), Y.Ulrich (Liverpool), D. Radic (Zurich & PSI), L.Naterop (Zurich & PSI), M.Rocco (Turin)
not shown: not shown: F.Hagelstein (Mainz), N.Schalch (Oxford), P.Banerjee (Cosenza), M.Ronchi (Mainz), Y.Fang (PSI), G.Billis (PSI), J.Wilson (Liverpool)

McMule

mule-tools.gitlab.io