

Simon Eidelman School

Monte Carlo generators

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I hope to answer the following questions

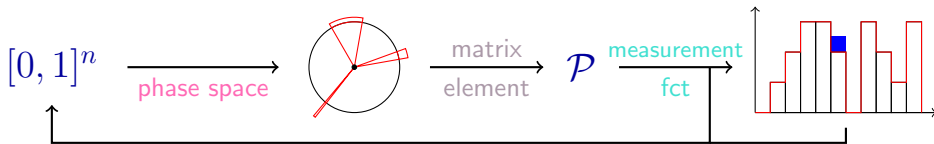
- why do we need MC generators?
- what changes beyond LO?
- what effects dominate? can we do better?
- what is resummation?

If you have questions, now or later, get in touch! yannick.ulrich@cern.ch

precision measurements

- we want to measure $a_{\mu}^{\text{HVP}} \sim \int_{4m_{\pi}^2}^{\infty} ds \, K(s) \times \sigma(ee \rightarrow \text{hadr})$
 - precise measurements $\Rightarrow$ precise modelling for acceptance, fits, etc.
 - analytic: possible but tedious & difficult
- $\Rightarrow$ numeric solution using Monte Carlo to sample / integrate over phase space
- to first approximation eg. for $ee \rightarrow \mu\mu$: $\sigma = \int d\Phi_{2 \rightarrow 2} |\mathcal{A}_{2 \rightarrow 2}^{(0)}|^2 \times S(\{p_i\})$

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- the measurement function: experimental cuts, rejecting or accepting events as needed
→ will come back to this

... is knowing σ enough though?

- for simulations we need actually generate events according to $d\mathcal{P} \sim d\Phi_{2 \rightarrow 2} |\mathcal{A}_{2 \rightarrow 2}^{(0)}|^2$

sampling from a p.d.f. $\mathcal{P}(\{p_i\})$ is generally difficult... $\Rightarrow$ hit-and-miss sampling

- generate a random event $\{p_i\}$ and calculate $\mathcal{P}(\{p_i\})$
- generate a random number $r \in [0, 1]$
- accept event if $r < \mathcal{P}(\{p_i\}) / \max \mathcal{P}$

this can be quite wasteful

- need to find $\max \mathcal{P}$ by 'training' Monte Carlo
- if average weight $\langle \mathcal{P} \rangle \ll \max \mathcal{P}$, most events will be rejected
- various ways of optimising this: vegas [Lepage 80], foam [Jadach 02] normalising flow (eg. [Gao, Isaacson, Krause 20]), ...

- LO is not very precise
- ⇒ just expand higher in perturbation theory
- this is what's called **fixed-order**
- best ever ($gg \rightarrow H$ [Anastasiou, Duhr, Dulat, Herzog, Mistlberger 15]): **NNLO**
- here: **NNLO** ($2 \rightarrow 2$) or **NLO** ($2 \rightarrow 3$) [Carloni Calame et al. 20; Banerjee, Engel, Signer, YU 20; Broggio et al 22]

$$\begin{aligned}
 \sigma = & \int d\Phi_{2 \rightarrow 2} \left| \begin{array}{c} \text{tree} \\ \text{1-loop} \\ \text{2-loop} \\ \text{3-loop} \\ \vdots \end{array} \right|^2 \\
 & + \int d\Phi_{2 \rightarrow 3} \left| \begin{array}{c} \text{1-loop} \\ \text{2-loop} \\ \text{3-loop} \\ \vdots \end{array} \right|^2 \\
 & + \int d\Phi_{2 \rightarrow 4} \left| \begin{array}{c} \text{2-loop} \\ \text{3-loop} \\ \vdots \end{array} \right|^2 \\
 & + \int d\Phi_{2 \rightarrow 5} \left| \begin{array}{c} \text{3-loop} \\ \vdots \end{array} \right|^2 \\
 & + \dots
 \end{aligned}$$

$$\sigma = \int d\Phi_{2 \rightarrow 2} \left| \text{diagram 1} + \text{diagram 2} \right|^2 \times S(\{p_i\}) + \int d\Phi_{2 \rightarrow 3} \left| \text{diagram 3} \right|^2 \times S(\{p_i, k\})$$

- **virtual matrix element:** momentum flowing through the loop unconstrained
 $\Rightarrow$ integrate over it
- **three-particle phase space:** emitted photon can be hard or soft (high or low energy)
 $\Rightarrow$ even below the detector resolution!
- **real matrix element:** $\sim \frac{1}{(p_e \cdot k)} \sim \frac{1}{E_\gamma} \frac{1}{(1 - \beta_e \cos \theta_{e\gamma})}$
- real and virtual are separately divergent but their sum is finite (KLN theorem)

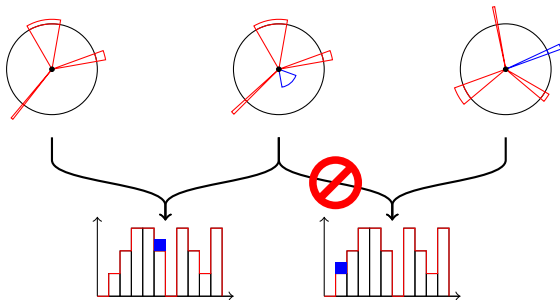
IR safety is not straightforward...

- measurement fct.: $S = 1$ to accept event, $S = 0$ rejects
- use this for histogramming $S \rightarrow S_1, \dots, S_n$ for n bins
- event may have more particles (such as photons)
- what if the event has a soft photon?

$\Rightarrow$ cuts must not change for $k \rightarrow 0$ (soft safety)

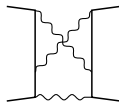
$\Rightarrow$ bin must also **not** change for $k \rightarrow 0$ (local soft safety)

- what about a collinear photon? (collinear safety)



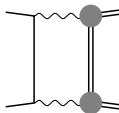
for leptons

- almost always done (semi)analytically
- can use decades of LHC tech, regulate by shifting $d \rightarrow 4 - 2\epsilon$
- fully automatised at one-loop (eg. [Buccioni et al. 19]), very difficult at two-loop
- bottleneck: calculation of so-called $\mathcal{O}(100)$ master integrals $\vec{I}(s_{ij})$
- currently favoured method: solve $\frac{\partial \vec{I}}{\partial s_{ij}} = M \vec{I}$, analytically or numerically (recent game changer: AMFlow [Liu, Ma, Wang 17])



for hadrons

- the counting is a bit muddled, unclear what NNLO would mean
- different approaches: sQED, $F \times$ sQED, FsQED, GVMD, (maybe) full hadronic model in the future



goal: regulate divergence & integrate real corrections numerically

$$\sigma_{\text{NLO}} = \int \left(\text{tree} + \frac{\alpha}{4\pi} \int \text{virt} \right) + \frac{\alpha}{4\pi} \int \text{real}$$

- slicing: approximate radiation below cut-off

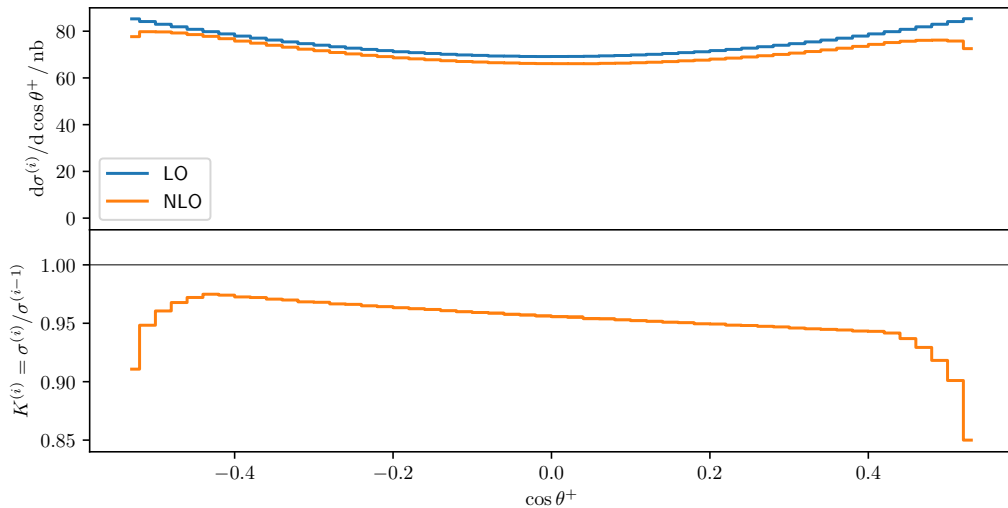
$$= \int \left(\text{tree} + \frac{\alpha}{4\pi} \text{virt} + \frac{\alpha}{4\pi} \int_{1, \omega < \omega_c} \text{real} \right) + \frac{\alpha}{4\pi} \int_{\omega > \omega_c} \text{real}$$

- subtraction: counter term to share singular structure

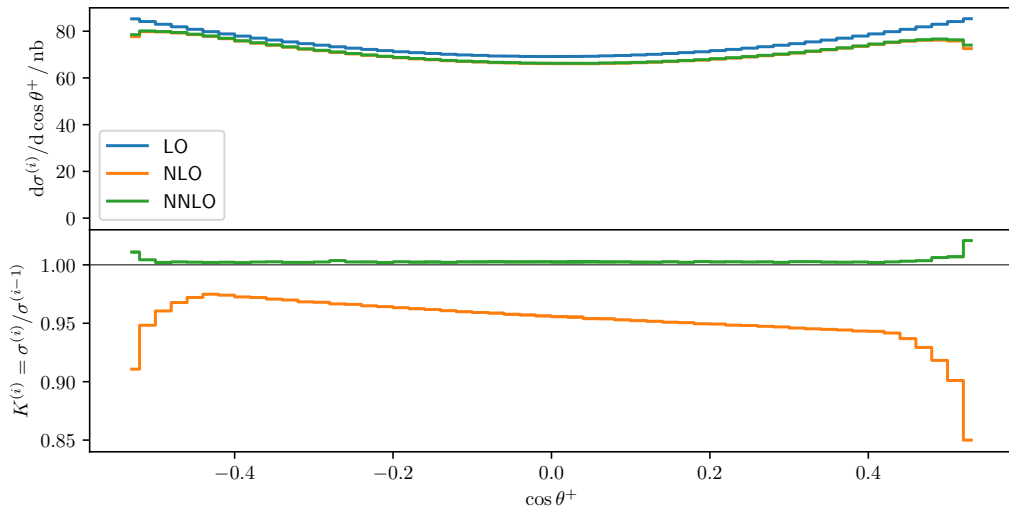
$$= \int \left(\text{tree} + \frac{\alpha}{4\pi} \text{virt} + \frac{\alpha}{4\pi} \int_1 \text{real} \right) + \frac{\alpha}{4\pi} \int \left(\text{real} - \text{real} \right)$$

this works but generation now becomes very tricky!

$ee \rightarrow \mu\mu$ CMD-ish ($\sqrt{s} = 700$ MeV)



$ee \rightarrow \mu\mu$ CMD-ish ($\sqrt{s} = 700$ MeV)



$\alpha = 1/137$ so why do we see 10% corrections?

$$\sigma \sim \int \text{[diagram of a circle with 6 external lines]} \sim \int d\Phi_{2 \rightarrow 3} \frac{1}{E_\gamma^2} \frac{1}{1 - \beta_e \cos \theta_{e\gamma}} \times S(\{p_i\})$$

- becomes large for $E_\gamma \rightarrow 0$ (even after slicing/subtraction) $\Rightarrow$ soft enhancement
- becomes large for $\theta_{e\gamma} \rightarrow 0$ because $\beta_e \sim 1 - 2m_e^2/s \Rightarrow$ collinear enhancement

$$\sigma \sim \frac{\alpha}{4\pi} \log \frac{1+\beta}{1-\beta} \log \frac{\omega_S^2}{s} \sim \frac{\alpha}{4\pi} \underbrace{\log \frac{m^2}{s}}_{L_m} \underbrace{\log \frac{\omega_S^2}{s}}_{L_s}$$

- ω_S depends on the measurement function $\Rightarrow L_s$ can be large but needn't be
- $L_m \sim 10$ almost always large

$\Rightarrow$ counting parameter is not $\alpha \sim 0.01$, it's $\alpha L \sim 0.1$!

don't expand in α , expand in αL

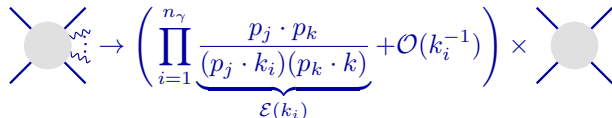
- best ever (EEC in $ee \rightarrow \text{jets}$, analytic, [Duhr, Mistlberger, Vita 22]): $N^4\text{LL}$
- numerically: NLL (eg. ALARIC [Herren et al. 23] & PanScales [van Beekveld et al. 22]), mostly LL(+) though
- our goal

$$\sigma = \sum_{n_\gamma=0}^{\infty} \int d\Phi_{2 \rightarrow 2+n_\gamma} \left| \sum_{\ell=0}^{\infty} \mathcal{A}_{2 \rightarrow 2+n_\gamma}^{(\ell)} \right|^2$$

- estimate $\mathcal{A}_{2 \rightarrow 2+n_\gamma} \approx X_{n_\gamma} \times \mathcal{A}_{2 \rightarrow 2}$ in some limit

$\sigma = \sigma_{0,0}$					LO
$+\left(\frac{\alpha}{\pi}\right)^1 L^1 \sigma_{1,1} + \left(\frac{\alpha}{\pi}\right)^1 L^0 \sigma_{1,0}$					NLO
$+\left(\frac{\alpha}{\pi}\right)^2 L^2 \sigma_{2,2} + \left(\frac{\alpha}{\pi}\right)^2 L^1 \sigma_{2,1} + \left(\frac{\alpha}{\pi}\right)^2 L^0 \sigma_{2,0}$					NNLO
$+\left(\frac{\alpha}{\pi}\right)^3 L^3 \sigma_{3,3} + \left(\frac{\alpha}{\pi}\right)^3 L^2 \sigma_{3,2} + \left(\frac{\alpha}{\pi}\right)^3 L^1 \sigma_{3,1} + \left(\frac{\alpha}{\pi}\right)^3 L^0 \sigma_{3,0}$					NNNLO
...					$N^n\text{LO}$
LL	NLL	NNLL	...	$N^n\text{LL}$	

- soft limit for any number of loops & photons $\rightarrow$ eikonal $\mathcal{E}$ [Yennie, Frautschi, Suura 61]



$$\rightarrow \left(\prod_{i=1}^{n_\gamma} \underbrace{\frac{p_j \cdot p_k}{(p_j \cdot k_i)(p_k \cdot k)}}_{\mathcal{E}(k_i)} + \mathcal{O}(k_i^{-1}) \right) \times$$

- integration factorises, define $\hat{\mathcal{E}} = \int d\Phi_\gamma \mathcal{E}$
- similarly for virtual, define $\check{\mathcal{E}} = \int \frac{1}{k^2} \frac{(2p_j - k) \cdot (2p_k + k)}{(k^2 - 2k \cdot p_k)(k^2 + 2k \cdot p_k)}$
- YFS form factor $Y = \hat{\mathcal{E}} + \check{\mathcal{E}}$ is finite

approximate all photons eikonal

$$\sigma = \sum_{n_\gamma=0}^{\infty} \int d\Phi_{2 \rightarrow 2+n_\gamma} \underbrace{\frac{1}{n_\gamma!} e^{-(-Y)}}_{\text{Poisson}} \left(\prod_{i=1}^{n_\gamma} \mathcal{E}(k_i) \right) |\mathcal{A}_{2 \rightarrow 2}^{(0)}|^2$$

procedure:

- generate undressed event
- sample from Poisson distribution with $\langle n_\gamma \rangle \approx -Y$
- generate i -th photons according to $\mathcal{E}(k_i)$
- regenerate rest of event
- fix mistakes

- consider an off-shell electron $p^2 \neq m^2$ with energy E
- splitting function $P \equiv P_{ee}(z)$: probability of emitting a collinear photon s.t. energy is now $E_1 = zE$

$$\begin{array}{c} \diagup \\ \text{---} \bigcirc \text{---} \\ \diagdown \end{array} \begin{array}{c} \text{---} \text{---} \\ \text{---} \end{array} \rightarrow \frac{\alpha}{2\pi} \frac{1}{p^2} \underbrace{\frac{1+z^2}{1-z}}_{P_{ee}(z)} \begin{array}{c} \diagup \\ \text{---} \bigcirc \text{---} \\ \diagdown \end{array} \begin{array}{c} \text{---} \text{---} \\ \text{---} \end{array} \left(q \rightarrow q/z \right)$$

- probability of radiating photon $E > z_{\min} \times E$: $R(p^2) = \frac{\alpha}{2\pi} \frac{1}{p^2} \int_0^{1-z_{\min}} dz P_{ee}(z)$
- Sudakov factor $\Delta(s_1, s_2)$: probability of no resolvable radiation between $p^2 = s_1$ and $p^2 = s_2$

$$\Delta(s_1, s_2 + \delta s) = \Delta(s_1, s_2) \times \left(1 - \int_{s_2}^{s_2 + \delta s} ds R(s) \right)$$

Markov process

- start with some virtuality, eg. $s_1 = st/u$ [Carloni Calame et al. 00]
- find next virtuality s_{i+1} from $\Delta(s_i, s_{i+1}) = r \in [0, 1]$
- find momentum fraction z , distributed according to $\mathcal{P} = \frac{\alpha}{2\pi} P(z)$
- keep doing this until $s_n = m^2$

modelling angular distribution [Carloni Calame 01]

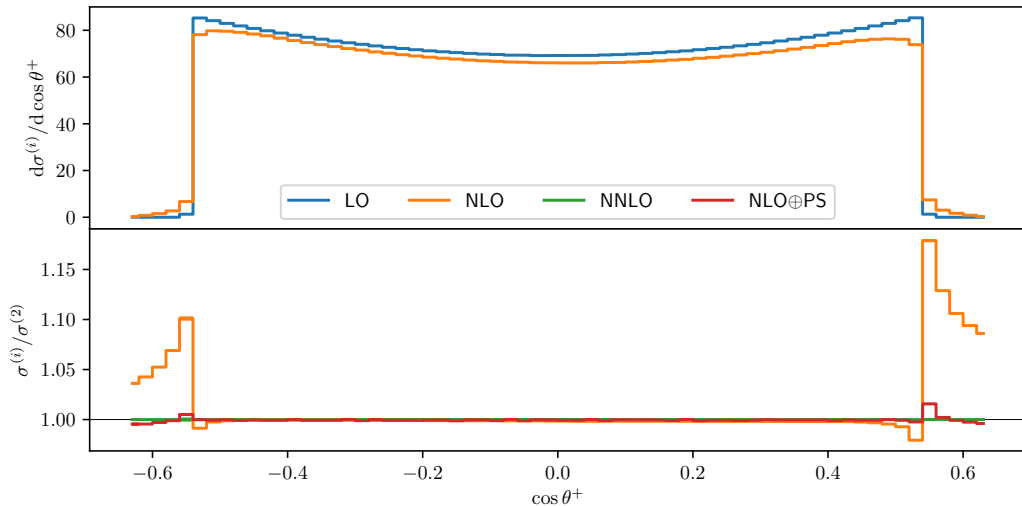
- this just generates collinear photons
- can be improved by samplings angles from $\mathcal{E}$

... sadly, not easily

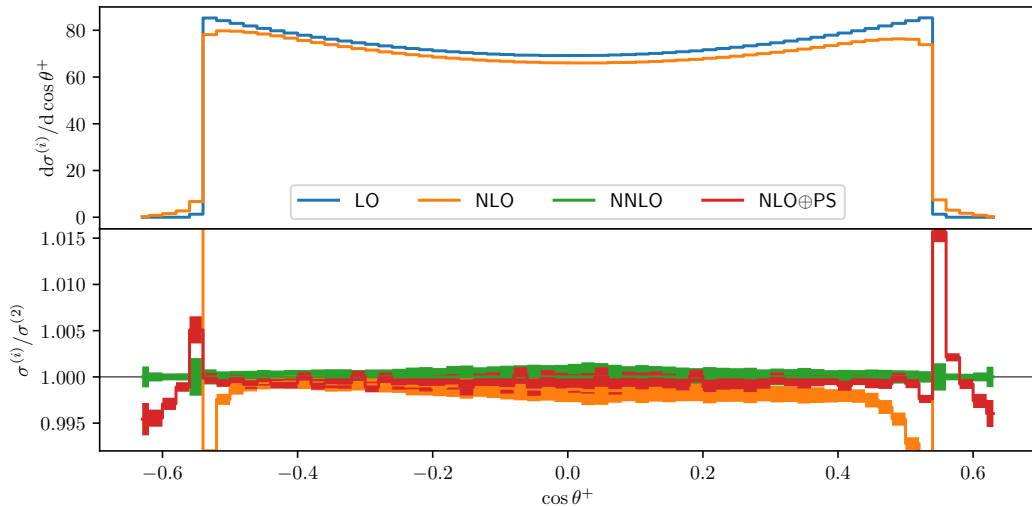
- naively doing all of this (FO + YFS + PS) will result in double counting
- $\text{FO} \oplus \text{YFS}$ is easy to any order in α , just use the actual matrix element rather than $\mathcal{E}$
- $\text{FO} \oplus \text{PS}$: understood at NLO [Balossini et al. 06], beyond WIP (“matching”)
- “ $\text{PS} \oplus \text{YFS}$ ”: can use $\mathcal{E}$ for angles in PS or P_{ee} for matrix elements in YFS

- for finiteness, only photon radiation is required
- what about pair production $ee \rightarrow XX + ee, \mu\mu, \pi\pi, \tau\tau, \dots?$
- can be physically separated as long as $m_f > 0$
- might be kinematically suppressed / impossible, depending on s
- but: corrections $\sim \log \frac{m_f^2}{s}!$
- combining with corresponding virtual actually reduces corrections for $f = e$
- possible to capture efficiently using eg.
 - PS with other splittings such as $P_{e\gamma} (\gamma \rightarrow ee)$
 - YFS (can be viewed as $\text{YFS} \oplus \text{PS}$) [Flower, Schönherr 22]

$ee \rightarrow \mu\mu$ CMD-ish ($\sqrt{s} = 700$ MeV) [FO: McMule, PS: BabaYaga@NLO]



$ee \rightarrow \mu\mu$ CMD-ish ($\sqrt{s} = 700$ MeV) [FO: McMule, PS: BabaYaga@NLO]



- in QCD: four-pt scale variation (full result does not depend on $\mu \rightarrow$ theory error $\approx$ scale dependence)
 - doesn't really work for QED
 - compare codes that do different things (assuming correctness & validity)
 - all PS have the same LL but different parts of the NLL
 - FO vs. truncated PS: missing higher order
 - PS vs. FO: dominant effects in PS
 - all of this is really tricky & difficult...
- $\Rightarrow$ most codes will **not** give you a theory error, just a statistical error (that may very well be underestimated!)

in the end, the only way to estimate missing effects is to calculate them and note their size

- MC are very important
- choice of tool non-trivial, depends on situation
- ideally compare multiple codes and see what matters for your experiment
- advertisements
 - I work on a FO-NNLO code called McMule, YFS WIP (LL, maybe even NLP)
<https://mule-tools.gitlab.io>
 - RadioMonteCarLow 2: effort to provide better codes to $ee \rightarrow \text{stuff}$ at $s \sim (\text{few GeV})^2$.
<https://radiomontecarlow2.gitlab.io>

a non-exhaustive list of resource I've used for this

- [RadioMonteCarlow 2 review \(draft online now\)](#)
- structure functions (analytic resummation in QED) [[Beenakker, Berends, van der Marck 90](#)]
- initial state QED for e^+e^- review [[Snowmass 22](#)]
- broad generators for high-energy physics review [[Snowmass 22](#)]
- review of state-of-the-art for μ - e scattering [[Banerjee et al. 20](#)]
- Parton showers for QED [[Carloni Calame 01](#)]
- various [diploma](#) & [PhD theses](#)
- Books: [QCD and Collider Physics](#) & [The Black Book of Quantum Chromodynamics](#)
- Tasi lecture on parton shower event generators [[Höche 14](#)]

here: at tree-level, full proof in modern notation see eg. [Engel 22]

$$\begin{aligned}
 \mathcal{A}_{n+1}^{(0)} &= \underbrace{\text{diagram with wavy line}}_{\text{subleading}} + \sum_{\text{leg}} \text{diagram with curly line} = \sum_i \left(\Gamma_i(p_i - k) \frac{\not{p}_i - \not{k} + m}{(p_i - k)^2 - m_i^2} \gamma^\mu u(p_i) \right) \epsilon^\mu + \mathcal{O}(k^0) \\
 &\stackrel{*}{=} \sum_i \left(\Gamma_i(p_i) \frac{\not{p}_i + m_i}{2p_i \cdot k} \gamma^\mu u(p_i) \right) \epsilon^\mu + \mathcal{O}(k^0) \\
 &\stackrel{\dagger}{=} \sum_i \left(\Gamma(p) \frac{2p^\mu + \gamma^\mu (-\not{p}_i + m_i)}{2p_i \cdot k} u(p_i) \right) \epsilon^\mu + \mathcal{O}(k^0) \\
 &\stackrel{\#}{=} \text{diagram with cross} \times \sum_i \frac{p_i \cdot \epsilon}{p_i \cdot k} + \mathcal{O}(k^0)
 \end{aligned}$$

* drop terms k^0 , $\dagger$ anticommute $\{\not{p}_i, \gamma^\mu\} = 2p_i^\mu$, $\#$ Dirac equation $(\not{p} - m)u = 0$

when generating we've made the following mistakes, reweight as follows

- used the undressed q_j eikonals to generate photons instead of dressed p_j :

$$W_{\text{dipole}} = \prod_i \frac{\mathcal{E}(\{p_j\}, k_j)}{\mathcal{E}(\{q_j\}, k_j)}$$

- used e^{-n_γ} instead of n^Y when generating n_γ : $W_{\text{YFS}} = \exp(Y + \langle n \rangle)$
- depending on how event is generated, other corrections might be needed as well (Lorentz boost, momentum mapping etc.)

angular distribution

- we need to sample θ and ϕ according to $\mathcal{E}$
- common strategy: pick one dipole of particles i and j and sample θ in dipole frame according to $\mathcal{E}_{ij} = \frac{p_i \cdot p_j}{p_i \cdot k p_j \cdot k}$

- probability of radiating: $R(p^2) = \frac{\alpha}{2\pi} \frac{1}{p^2} \int_0^{1-z_{\min}} dz P_{ee}(z)$
- probability of not radiating between s_1 and s_2 : $\Delta(s_1, s_2)$ (per definition)
- going a small step further $s_2 \rightarrow s_2 + \delta s$: probabilities multiply

$$\Delta(s_1, s_2 + \delta s) = \underbrace{\Delta(s_1, s_2)}_{[s_1, s_2]} \times \underbrace{\left(1 - \int_{s_2}^{s_2 + \delta s} ds R(s)\right)}_{[s_2, s_2 + \delta s]} \approx \Delta(s_1, s_2) - \Delta(s_1, s_2) \delta s R(s_2)$$

- this is a differential equation $\frac{d\Delta(s_1, s_2)}{ds_2} = -R(s_2)\Delta(s_1, s_2)$
- solved by $\Delta(s_1, s_2) = \exp\left(-\int_{s_1}^{s_2} \frac{ds}{s} \int_0^{1-z_{\min}} dz \frac{\alpha}{2\pi} P_{ee}(z)\right)$