Paul Scherrer Institut · Universität Zürich

3rd workstop

"paving the way to alternative NNLO strategies"

The FKS^2 subtraction scheme: status and future

Yannick Ulrich

for the McMule collaboration

(P. Banerjee, T. Engel, A. Signer, YU)

Paul Scherrer Institut / Universität Zürich

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what this is and is not

- this is us asking for help, not us telling you the solution

$\Rightarrow$ this is not a conference talk, everything is in flux

- in the spirit of a workstop

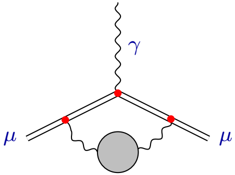
what I will talk about

- explain massification
- explain FKS and FKS^2
- state the problems
  - numerical finite matrix elements in $d = 4$
  - collinear stabilisation / massification

- measure HVP using t-channel μ-e scattering at 10ppm

$$a_\mu^{\mathsf{HLO}} = \frac{\alpha}{\pi}\int_0^1 \mathrm{d}x(1-x)\underbrace{\Delta\alpha_{\mathsf{had}}\Big(\frac{x^2}{x-1}m_\mu^2\Big)}_{\propto \mathrm{d}\sigma/\mathrm{d}t}$$

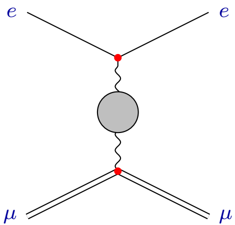
[Carloni Calame et al. 2015, Abbiendi et al 2016]



- proposed experiment @ CERN's M2

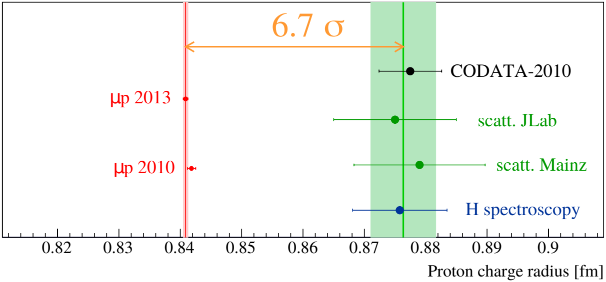
[Matteuzzi, YU et al 2019 (LOI)]

- huge theory effort [Alacevich et al 2018, Mastrolia et al 2017 and 2018, Fael, Passera 2019]



⇒ NNLO emission-from-e-line-only (largest part) [Banerjee, Engel, Signer, YU soon] using [Bernreuther et al. 2004] with FKS^2

Proton radius puzzle: 6σ-discrepancy between μp and ep



[A. Antognini, PSI 2019]

- re-measure ep scattering @ JLab (PRad)

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{ep} = \epsilon_{\text{exp}} \times \frac{N(ep \to ep \text{ in } \theta)}{N(ee \to ee)} \times \underbrace{\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{ee}}_{\text{main uncertainty}}$$

environment

what we want

- parton-level, NNLO (and beyond) (fully) massive calculations

⇒ fully differential, no analytic integration !

- MEG, MUonE, MUSE, MESA, Prad, ...

what we have

- a way to obtain leading mass effects in any two-loop amplitude (massification)
- an NNLO subtraction scheme for massive QED (FKS^2)
- the framework to implement many processes (McMule)

massification /ˈmasɪˈfɪkeɪʃ(ə)n/

(noun)

Methods to add leading mass effects to amplitudes in perturbation theory

Derivatives:

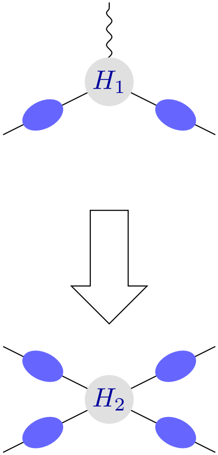
- **massify** verb
- **massified** adjective

massification: brief history C_F

one external mass $m \ll Q^2 = s$

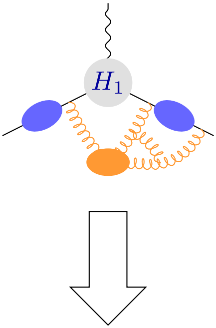
- SCET inspired $\sim$ fragmentation fct.
- Bhabha scattering (photonic) [Penin 2005]

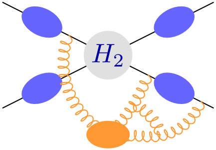
  matching $1/\epsilon \to \ln m_e, \ln m_\gamma$
- QCD with $n_f = n_m = n_h = 0$ [Mitov, Moch 2006]
- matching $F_{\gamma^*}(m) = \sqrt{Z_J} \times \sqrt{Z_J} \times F(0)$
- Z_J: jet fct., independent of hard scale s, $\supset \ln(m^2/\mu^2)$



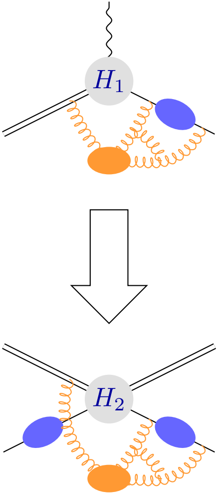
light fermion loops

- n_m terms [Becher, Melnikov 07]
- $F_{\gamma^*}(m) = \mathcal{S} \times \sqrt{Z_J} \times \sqrt{Z_J} \times F(0)$
- $\mathcal{S}(s,m)$: soft function, only contributions from vacuum polarization diagrams with massive fermions, $\supset \ln(m^2/s)$
- $\mathcal{A}_{ee\to ee}(m) = \mathcal{S}' \times Z_J^{4/2} \times \mathcal{A}(0)$
- factorisation $\leftrightarrow$ resummation via RG equations





- two different masses $M \gg m \gg 0$ [Engel, Gnendiger, Signer, YU 18]
- $F_\mu(m) = \mathcal{S} \times \sqrt{Z_q} \times F_\mu(0)$
- $\mathcal{A}_{\mu e}(m) = \mathcal{S}' \times \sqrt{Z_q \times \bar{Z}_q} \times \mathcal{A}(0)$



The FKS^2 scheme:

status

- originally developed for QCD [Frixione, Kunszt, Signer 1995]
- everything massive $\rightarrow$ no collinear singularities

$\Rightarrow$ most complexity drops out

credo at NLO and beyond

- everything that vegas sees needs to be finite (explicitly and implicitly)

$\Rightarrow$ keep subtracting stuff until it is

- example: $\mu(p) \to \nu\bar{\nu}e(q) + \gamma(k)$

$$\lim_{\xi\to 0} \mathcal{M}^{(0)}_{n+1} = \mathcal{M}^{(0)}_{n}\mathcal{E} \propto \frac{1}{(p\cdot k)(q\cdot k)} \propto \frac{1}{\xi^2}\frac{1}{1-y\beta}$$

- with $\xi \propto k^0$, $\cos\sphericalangle(q,k) = y$, and $\beta = |\vec{q}|/q^0$
- write $\mathrm{d}\sigma_r$

$$\mathrm{d}\sigma_r = \underbrace{\mathrm{d}\phi_{n+1}}_{\propto\xi^{1-2\epsilon}\mathrm{d}\xi} \underbrace{\mathcal{M}^{(0)}_{n+1}}_{\supset\xi^{-2}} \propto \mathrm{d}\phi_n \times \frac{\mathrm{d}y\mathrm{d}\Omega^{(2-2\epsilon)}}{(1-y^2)^\epsilon}$$

$$\mathrm{d}\xi\,\xi^{-1-2\epsilon}\,\underbrace{\left(\xi^2\mathcal{M}^{(0)}_{n+1}\right)}_{\text{finite for }\xi\to 0}$$

FKS at NLO (cont.)

- introduce unphysical ξ_c and expand as c-distribution

$$\xi^{-1-2\epsilon} = \frac{\xi_c^{-2\epsilon}}{2\epsilon}\delta(\xi) + \Big(\xi^{-1-2\epsilon}\Big)_c$$

$$\int \mathrm{d}\xi \, \left(\frac{1}{\xi^n}\right)_c f(\xi) = \int \mathrm{d}\xi \, \frac{f(\xi) - f(0)\Theta(\xi-\xi_c)}{\xi^n}$$

- $\mathrm{d}\sigma_r = \mathrm{d}\sigma^{(s)} + \mathrm{d}\sigma^{(h)}$

$$\mathrm{d}\sigma^{(s)} \propto \mathrm{d}\phi_n \mathcal{M}_n^{(0)} \times \underbrace{\frac{\xi_c^{-2\epsilon}}{2\epsilon}\int \frac{\mathrm{d}y\mathrm{d}\Omega^{(2-2\epsilon)}}{(1-y^2)^\epsilon}\mathcal{E}}_{\hat{\mathcal{E}}(\xi_c)}$$

$$\mathrm{d}\sigma^{(h)} \propto \mathrm{d}\phi_n^{d=4}\mathrm{d}\xi\mathrm{d}y\mathrm{d}\Omega^{(2)} \times \left(\frac{1}{\xi}\right)_c \big(\xi^2\mathcal{M}_{n+1}^{(0)}\big)$$

FKS at NLO (cont.)

- $\mathrm{d}\sigma^{(h)}$ is finite $\Rightarrow$ $\epsilon\to 0$ numerically
- $\mathrm{d}\sigma^{(s)}$ is 'trivial' because of $\delta(\xi)$

$$\mathrm{d}\sigma^{(s)} \propto \frac{\xi_c^{-2\epsilon}}{2\epsilon}\mathcal{M}_n^{(0)} \int \mathrm{d}\Omega\, \mathcal{E} = \hat{\mathcal{E}}(\xi_c)\mathcal{M}_n^{(0)}$$

- eikonal $\mathcal{E}$ and $\hat{\mathcal{E}}$ is build from building blocks
- $\hat{\mathcal{E}}\mathcal{M}_n^{(0)} + \mathcal{M}_n^{(1)} =$ finite (KLN)
- use ξ_c to test implementation (any IR safe $\frac{\mathrm{d}^n\sigma}{\mathrm{d}x_1\cdots\mathrm{d}x_n}$ can't depend on ξ_c)

problems at NNLO

- matrix elements may not finite any more (explicit $1/\epsilon$)
- more ways photons can become soft

recipe to extend this to NNLO:

1$^{\text{st}}$ step everything vegas sees needs to be finite!

2$^{\text{nd}}$ step keep subtracting until it is

3$^{\text{rd}}$ step hope for the best and deal with fall-out later

$\mathbf{FKS^2}$: real-virtual

term 1: $\mathcal{M}^{(1)}_{n+1}$ (real × virtual)

- like NLO, but $\mathrm{d}\sigma^{(h)} \propto \mathcal{M}^{(1)}_{n+1}$ is not finite $\Rightarrow$ split further
  1. chop the pole (and induced terms): $\overline{\text{MS}}$-like subtraction
  2. $\mathcal{M}^{(1)}_{n+1} = \underbrace{\mathcal{M}^{(1)f}_{n+1}}_{\Rightarrow \mathrm{d}\sigma^{(fin)}} \underbrace{-\hat{\mathcal{E}}(\xi_{c_B})\mathcal{M}^{(0)}_{n+1}}_{\Rightarrow \mathrm{d}\sigma^{(sin)}}$: eikonal subtraction

  $\mathcal{M}^{(1)f}_{n+1}$ is finite (KLN of the $n+1$ process)
- $\mathrm{d}\sigma^{(fin)}$ and $\mathrm{d}\sigma^{(sin)}$ depend on two a-priori different ξ_{c_i}
- $\mathrm{d}\sigma^{(fin)}$ is now finite $\Rightarrow$ $\epsilon \to 0$
- we hope for the best and deal with $\mathcal{I}$ later

$$\mathrm{d}\sigma^{(sin)} \equiv -\mathcal{I}(\xi_{c_A}, \xi_{c_B}) \propto \int_{\xi_{c_A}} \hat{\mathcal{E}}(\xi_{c_B})\ (\xi^2 \mathcal{M}^{(0)}_{n+1})$$

FKS^2: real-real

term 2: $\mathcal{M}^{(0)}_{n+2}$ (real × real)

- do normal FKS twice with two ξ_c, but (h) mixes with (s)

$$\mathrm{d}\sigma_{n+2} = \underbrace{\mathrm{d}\sigma^{(hh)}}_{\epsilon\to 0} + \underbrace{\mathrm{d}\sigma^{(ss)}}_{\hat{\mathcal{E}}(\xi_{c_1})\,\hat{\mathcal{E}}(\xi_{c_2})} + \mathrm{d}\sigma^{(hs)} + \mathrm{d}\sigma^{(sh)}$$

- mixed terms $\mathrm{d}\sigma^{(hs)}$ are troublesome

$$\mathrm{d}\sigma^{(hs)} \propto \frac{1}{2!}\int_{\xi_{c_1}} \hat{\mathcal{E}}(\xi_{c_2})\ (\xi^2\mathcal{M}^{(0)}_{n+1}) = \frac{1}{2!}\mathcal{I}(\xi_{c_1}, \xi_{c_2})$$

- luckily $\mathrm{d}\sigma^{(aux)} \equiv \mathrm{d}\sigma^{(sin)} + \mathrm{d}\sigma^{(hs)} + \mathrm{d}\sigma^{(sh)} = 0$ if all ξ_{c_i} are equal

we now have

$$\mathrm{d}\sigma_n = \mathrm{d}\phi_n^{d=4} \underbrace{\Big(\mathcal{M}_n^{(2)} + \hat{\mathcal{E}}(\xi_c)\mathcal{M}_n^{(1)} + \frac{1}{2!}\hat{\mathcal{E}}(\xi_c)^2\mathcal{M}_n^{(0)}\Big)}_{\mathcal{M}_n^{(2)f}(\xi_c)}$$

$$\mathrm{d}\sigma_{n+1} = \frac{1}{1!}\mathrm{d}\varphi_{n+1}^{d=4}\ \Big(\frac{1}{\xi}\Big)_c \Big(\xi^2\mathcal{M}_{n+1}^{(1)f}(\xi_c)\Big)$$

$$\mathrm{d}\sigma_{n+2} = \frac{1}{2!}\mathrm{d}\varphi_{n+2}^{d=4}\ \Big(\frac{1}{\xi_1}\Big)_c \Big(\frac{1}{\xi_2}\Big)_c \Big(\xi_1^2\xi_2^2\mathcal{M}_{n+2}^{(0)f}\Big)$$

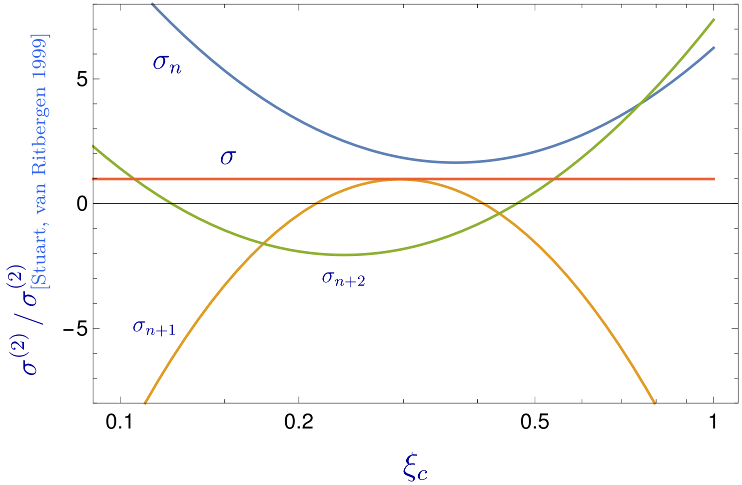
where $\mathrm{d}\varphi_n$ is the n-particle phasespace without the $\xi^{1-2\epsilon}$ factors

$\mathbf{FKS^2}$: summary (cont.)

very simple NNLO scheme!

- no complicated analytic integrals
- no ϵ/ϵ-terms
- $\mathcal{M}_n^{(\ell)f}$ is scale- and regularisation-scheme independent
- obtain $\mathcal{M}_n^{(2)}$ using massification
- $\mathcal{O}(\epsilon)$ terms of $\hat{\mathcal{E}}$ vanish exactly
- down-side: phasespace parametrisation is non-trivial

ξ_c independent!



many more terms at N^{3}LO!

$$\mathrm{d}\sigma_n^{(3)}(\xi_c) = \mathrm{d}\Phi_n\Big(\mathcal{M}_n^{(3)} + \underbrace{\hat{\mathcal{E}}(\xi_c)\mathcal{M}_n^{(2)}}_{\mathrm{d}\sigma_s^{(3)}} + \underbrace{\frac{1}{2!}\hat{\mathcal{E}}(\xi_c)^2\mathcal{M}_n^{(1)}}_{\mathrm{d}\sigma_{ss}^{(3)}} + \underbrace{1\times\frac{1}{3!}\hat{\mathcal{E}}(\xi_c)^3\mathcal{M}_n^{(0)}}_{\mathrm{d}\sigma_{sss}^{(3)}}\Big)$$

$$\overbrace{\underbrace{-\mathcal{I}(\xi_c) - \frac{1}{2!}\mathcal{J}(\xi_c)}_{\mathrm{d}\sigma_d^{(3)}} + \underbrace{\frac{1}{2!}\mathcal{I}(\xi_c) + \frac{1}{2!}\mathcal{I}(\xi_c)}_{\mathrm{d}\sigma_{hs}^{(3)}+\mathrm{d}\sigma_{sh}^{(3)}} \underbrace{-\frac{1}{2!}\mathcal{K}(\xi_c)}_{\mathrm{d}\sigma_{hd}^{(3)}} + \underbrace{3\times\frac{1}{3!}\mathcal{J}(\xi_c)}_{\mathrm{d}\sigma_{hss}^{(3)}+\cdots} + \underbrace{3\times\frac{1}{3!}\mathcal{K}(\xi_c)}_{\mathrm{d}\sigma_{hhs}^{(3)}+\cdots}}^{0}$$

$$\mathrm{d}\sigma_{n+1}^{(3)}(\xi_c) = \mathrm{d}\sigma_f^{(3)} = \tfrac{1}{1!}\mathrm{d}\varphi_{n+1}\,\left(\tfrac{1}{\xi}\right)_c\Big(\xi^2\mathcal{M}_{n+1}^{(2)f}(\xi_c)\Big)$$

$$\mathrm{d}\sigma_{n+2}^{(3)}(\xi_c) = \mathrm{d}\sigma_{hf}^{(3)} = \tfrac{1}{2!}\mathrm{d}\varphi_{n+2}\,\left(\tfrac{1}{\xi_1}\right)_c\left(\tfrac{1}{\xi_2}\right)_c\Big(\xi_1^2\xi_2^2\mathcal{M}_{n+2}^{(1)f}\Big)$$

$$\mathrm{d}\sigma_{n+3}^{(3)}(\xi_c) = \mathrm{d}\sigma_{hhh}^{(3)} = \tfrac{1}{3!}\mathrm{d}\varphi_{n+3}\,\left(\tfrac{1}{\xi_1}\right)_c\left(\tfrac{1}{\xi_2}\right)_c\left(\tfrac{1}{\xi_3}\right)_c\Big(\xi_1^2\xi_2^2\xi_3^2\mathcal{M}_{n+3}^{(0)f}\Big)$$

with $\mathcal{M}_m^{(1)f}$ as above and $\mathcal{M}_m^{(2)f} = \mathcal{M}_m^{(2)} + \frac{\hat{\mathcal{E}}^1}{1!}\mathcal{M}_m^{(1)} + \frac{\hat{\mathcal{E}}^2}{2!}\mathcal{M}_m^{(2)}$

- there seems to be a pattern here

⇒ postulate behaviour at $\mathrm{N}^\ell\mathrm{LO}$

$$\mathrm{d}\sigma^{(\ell)}_{n+j} = \mathrm{d}\varphi^{d=4}_{n+j} \frac{1}{j!} \Big(\frac{1}{\xi_1}\Big)_c \cdots \Big(\frac{1}{\xi_j}\Big)_c \xi_1^2 \cdots \xi_j^2 \; \mathcal{M}^{(\ell-j)f}_{n+j}(\xi_c)$$

$$\mathcal{M}^{(\ell)f}_m = \sum_{j=0}^{\ell} \frac{\hat{\mathcal{E}}^j}{j!} \mathcal{M}^{(\ell-j)}_m$$

The FKS^2 scheme:

future

collinear stabilisation

mass hierarchy $m \ll M, \sqrt{s}, .. \to \; \beta \sim 1$

- integrand $\propto \frac{1}{1-y\beta}$ has a pseudo-singularity at $y = 1$
- integrable but annoying. so far: increase statistics
- find a suitable counter-term to remedy this (cf. [Dittmaier 1999])
- but don't destroy parton-level picture !
- this is different from QCD where collinear emission is always unresolved

- f-matrix elements (or eikonal-subtracted matrix elements) are central

$$\mathcal{M}_n^{(1)f} = \mathcal{M}_n^{(1)} + \hat{\mathcal{E}}(\xi_c)\mathcal{M}_n^{(0)} = \text{finite}$$

- can we calculate these objects directly in $d=4$?
- simplest example with $p^2 = q^2 = m^2$

$$\mathcal{M}_n^{(1)f}(\xi_c) = \Bigg(\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2}\frac{1}{k^2-2k\cdot p}\frac{1}{k^2-2k\cdot q}\Bigg) + \Bigg(\int \frac{\mathrm{d}^d k}{(2\pi)^d}\delta(k^2)\delta(k^0)\frac{\xi_c^{-2\epsilon}}{2\epsilon}\frac{\big(k^0\big)^2}{\big(k^2-2k\cdot p\big)\big(k^2-2k\cdot q\big)}\Bigg)$$

numerical f-matrix elements (cont.)

$$\mathcal{M}_n^{(1)f}(1) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - 2k\cdot p} \frac{1}{k^2 - 2k\cdot q} \left(\frac{1}{k^2 + \mathrm{i}0^+} + \delta(k^2)\delta(k^0)\frac{(k^0)^2}{2\epsilon} \right)$$

- idea: use LTD on $\frac{1}{k^2+\mathrm{i}0^+} = \frac{1}{k^2 - \mathrm{i}0^+ k^0} - \Theta(k^0)\delta(k^2)$

$$= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2 - 2k\cdot p} \frac{1}{k^2 - 2k\cdot q} \frac{1}{k^2 - \mathrm{i}0^+ k^0} + \int \mathrm{d}\phi \frac{1}{-2k\cdot p} \frac{1}{-2k\cdot q} \left(\frac{(k^0)^2}{2\epsilon}\delta(k^0) - 1 \right)$$

- does any of this make sense?
- how do I calculate these things numerically?